DECIDABILITY RESULTS IN NON-CLASSICAL LOGIC. III.
(Systems with Statability Operators)

Dedicated to the Memory of A. N. Prior

BY

DOV M. GABBAY

ABSTRACT

A Kripke type semantics is given to a large class of tense logics with statability operators (including Priors QK_t) in such a manner as to obtain their decidability using Rabin's theorem.

0. Introduction

In this paper we continue our applications of a theorem of M. O. Rabin and obtain decidability results for various tense and modal systems with statability operators. We assume familiarity with the methods of Part I of this paper.

Let us now survey briefly the results. Prior [3] considered a system QK_t which is obtained from Lemmon's K_t by adjoining the two unary operators $T\phi$ and $Y\phi$ with suitable axioms. $T\phi$ reads: ϕ is statable in all future possible worlds and $Y\phi$ reads: ϕ is statable in all past possible worlds. Rennie [5] gave Kripke type semantics to a modified version of QK_t, we shall call his system RK_t. Bull [1] also considered modal systems with statablity operators and propositional quantifiers.

In Section 1 we shall give, using Rennie [5] and our [2] methods, semantics for Prior's system QK_t and other systems weaker than those considered by Rennie. Our completeness proofs and semantics shall be given in such a manner as to enable us to prove decidability later on. In Section 2 we shall describe basic statability systems in which all other systems can be faithfully interpreted. In

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Section 3 we shall prove, using methods of [4] and [2] that the various systems considered are decidable.

1. The System SK_t and Extensions

We begin by describing the basic system SK_t, which is the weakest system we shall consider. Our language contains, besides the classical connectives the two tense operators $G\phi$ and $H\phi$ ($G\phi$ reads: ϕ is true in all relevant future possible worlds and $H\phi$ reads: ϕ is true in all relevant past possible worlds) and the two statability operators $T\phi$ and $Y\phi$. Before we give the axioms let us remark that in all the systems considered in this section $T\phi$ (or $Y\phi$) cannot have a truth value in a world x if ϕ is not statable at x; even though $T\phi$ relates to the statability of ϕ in the worlds other than x, namely those in the future of x.

2. The System SK_t

(1) Classical propositional tautologies and classical rules of inference.

(2) $T\phi \to Tp$ where p is a propositional variable occurring in ϕ and similarly $Y\phi \to Yp$.

(3) $(\bigwedge_i Tp) \to T\phi$
$(\bigwedge_i Yp_i) \to T\phi$

where $p_1 \cdots$ are all the propositional variables occurring in ϕ.

(4) $\sim \phi \to G \sim H\phi$
$\sim \phi \to H \sim G\phi.$

(5) $\vdash \phi \Rightarrow \vdash G\phi$ and $\vdash H\phi$.

(6) $T(\bigwedge_i p_i) \Rightarrow [G(\phi \to \psi) \to (G\phi \to G\psi)]$
$Y(\bigwedge_i p_i) \to [H(\phi \to \psi) \to (H\phi \to H\psi)]$
where $p_1 \cdots$ are all the propositional variables appearing in ϕ and not in ψ.

Prior's system QK_t is the extension of SK_t with axiom (7).

(7) $\sim T\phi \to H \sim T\phi$
$\sim Y\phi \to G \sim Y\phi.$

Rennie's system which we shall call RK_t is the extension of SK_t with (8).

(8) $G \sim (\phi \to \phi) \to [T\phi \to T\psi]$
$H \sim (\phi \to \phi) \to [Y\phi \to Y\psi]$.

If we extend SK_t with (7), (8) and (9), we get the system QCR of Rennie and Prior, where

(9) $G\phi \to GG\phi,\ H\phi \to HH\phi$.

As we shall see in a later section QCR is complete for transitive world systems and is decidable.

We shall now, following ideas of Rennie, describe a semantics for SK_t, QK_t, RK_t and QCR. (The semantics for RK_t was given by Rennie [5].)

We firstly define the basic statability structure. Each system will have as its semantics all structures which are obtained from the basic statability structures in a certain manner, characteristic to the specific system. The construction of these structures may seem to the reader unnecessarily complicated, however we do need it all in the proof that these systems are decidable.

Given a language L a statability structure is a system $(A_x, L_x, <, >, R_<, R_>, 0)$ $x \in W$, $0 \in W$, where W is the set of possible worlds, A_x for $X \in W$ is a classical propositional structure built on the denumerable propositional language $L_x \subseteq L$, and $<$, $>$, $R_<$ and $R_>$ are four binary relations on W. We require the following properties and relations to hold:

(10) If we define $x\rho^*y$ as $x < yVx > yVxR_<yVxR_>y$ then $(W, \rho^*, 0)$ is a tree with successor relation ρ^* and root point 0, and for every $x \in W$ we have that $0\rho^{*n}x$ for some n.

(11) $(x < yVxR_<y) \Rightarrow L_x \supseteq L_y$
$(x > yVxR_>y) \Rightarrow L_x \supseteq L_y$.

(12) Define: $(x\rho_<y)$ iff $(x < yVxRy)$ and $(x\rho_>y)$ iff $(x > yVyRx)$ where xRy is $(xR_<yVyR_>x)$.
Let $\bar{R}_<$, $\bar{R}_>$, $\bar{R}$, $\bar{\rho}_<$, $\bar{\rho}_>$ be the transitive closures of $R_<$, $R_>$, R, $\rho_<$, $\rho_>$ respectively. Let $x\rho y$ be $x\rho_<yVy\rho_>x$ or equivalently $(x < y)\ V\ (xR_<y)\ V\ (y > x)\ V\ (yR_>x)$ and let $\bar{\rho}$ be the transitive closure of ρ.

Let $(A_x, L_x, R_G, R_H, <_T, >_Y, 0)$ $0 \in W$, $x \in W$ be a structure (with R_G, R_H, $<_T$, $>_Y$ binary relations), which is obtained from the statability structure described above in some way. (Different Logics have different ways: for example, we may take $R_G = R$ etc.)

We now turn to define for this structure the truth value of a sentence ϕ at a possible $x \in W$, denoted by $[\phi]_x$.

(13) For a propositional variable p of our language L we define $[p]_x = 1$ (truth) if $p \in L_x$ and A_x gives p the value 1 and $[p]_x = 0$ if $p \in L_x$ and A_x gives the value 0 and $[p]_x$ is undefined otherwise.

(14) $\sim \phi$, $G\phi$, $H\phi$, $T\phi$, $Y\phi$ (resp. $\phi \wedge \psi$ are defined iff ϕ (resp ϕ and ψ) are defined.

(15) In case the formulae below are defined at $x \in W$ then their value at x is computed as follows:

$[\phi \wedge \psi]_x = 1$ iff $[\phi]_x = 1$ and $[\psi]_x = 1$.
$[\sim\phi]_x = 1$ iff $[\phi]_x = 0$.
$[G\phi]_x = 1$ iff for all y such that xR_Gy, $[\phi]_y = 1$ if defined at y.
$[H\phi]_x = 1$ iff for all y such that xR_Hy, $[\phi]_y = 1$ if defined at y.
$[T\phi]_x = 1$ iff for all y such that $x <_T y$ ϕ is defined at y.
$[Y\phi]_x = 1$ iff for all y such that $x >_Y y$ ϕ is defined at y.
ϕ is said to hold in the model iff $[\phi]_0 = 1$.

THEOREM 16 (*completeness theorem*).

(17) SK_t is complete for all structures $(A_x, L_x, R_G, R_H, <_T, >_Y, 0)$ $x \in W$, $0 \in W$ such that

(a) xR_Gy iff xRy
(b) xR_Hy iff yRx
(c) $x <_T y$ iff $x\rho_< y$
(d) $x >_Y y$ iff $x\rho_> y$.

(18) QK_t is complete for all structures such that

(a) xR_Gy iff xRy
(b) xR_Hy iff yRx
(c) $x <_T y$ iff $\exists u_0 u_1[(u_0\bar{R}_> x) \wedge (u_0\bar{R}_< u_1) \wedge (u_1\rho_< y)]$.
u_0 and u_1 may be equal to x and y respectively.
(d) $x >_Y y$ iff $\exists u_0 u_1[(u_0\bar{R}_< x) \wedge (u_0\bar{R}_> u_1) \wedge (u_1\rho_> y)]$.
u_0 and u_1 may be equal to x and y respectively.

(19) Rennie [5] RQ_t is complete for all structures such that

(a) xR_Gy iff $x\rho y$
(b) xR_Hy iff $y\rho x$
(c) $x <_T y$ iff $x\rho y$

(d) $x >_Y y$ iff $y\rho x$.

(2) QCR is complete for all structures such that

(a) $xR_G y$ iff $x\bar{\rho}y$

(b) $xR_H y$ iff $y\bar{\rho}x$

(c) $x <_T y$ iff $x\bar{\rho}y$

(d) $x >_Y y$ iff $y\bar{\rho}x$.

(21) To get semantics for SK_t + axiom (9) replace R by $\bar{R}$ in (17a) and (17b).

(22) To get semantics for RQ_t + axiom (7) replace ρ by $\bar{\rho}$ in (19a) and (19b). Before turning to the proof of the completeness theorem, we need a series of lemmas.

LEMMA 23. *In any structure of* (19) *we cannot have that* $x\rho y \wedge y\rho z$ and ϕ *is defined at x and z and not defined at y, for any x, y,* $z \in W$ *and a sentence* ϕ.

PROOF. Assume that $x\rho y$ and $y\rho z$, then since $x\rho y$ we have that either $xR_< y \vee x < y$ in which case by definition $L_x \supseteq L_y$, or $yR_> x \vee y > x$ in which case $L_y \supseteq L_x$. Similarly for the case of $y\rho z$. Now since ϕ is defined at x and not at y we get that not $L_y \supseteq L_x$ and so we must have that $xR_< y \vee x < y$ holds.

Similarly we get that (since ϕ is defined at z and not at y) $L_y \not\supseteq L_z$ and so we get that $zR_> y \vee z > y$ holds. Now since (10) holds we get that in the tree W z is the predecessor of y, and x is the predecessor of y which is a contradiction.

LEMMA 24. *In* (18c) $<_T$ *may be equivalently defined as* $x <_T y$ iff $\exists u(x\bar{R}u \wedge u\rho_< y)$ *and similarly for* (18d) $x >_T y$ iff $\exists u(u\bar{R}x \wedge u\rho_> y)$.

PROOF. Verify that $x\bar{R}y$ iff $\exists u[u\bar{R}_> x \wedge u\bar{R}_< y]$.

Let Δ be a complete and consistent SK_t theory in the language L_Δ, and let $\sim G\phi \in \Delta$. Let Δ_0 be $\{\sim\phi\}$ U $\{\psi \mid G\psi \in \Delta \cap L_\Delta\phi\}$ where $L_\Delta\phi$ is the language built up from all propositional variables occurring in ϕ or in any $T\beta \in \Delta$.

LEMMA 25. Δ_0 *is* SK_t *consistent.*

PROOF Otherwise for some $\psi_1 \cdots \psi_n$ we have $\vdash \psi_1 \wedge \cdots \wedge \psi_n \to \phi$ ($n > 0$ for otherwise $\vdash \phi$ and so $\vdash G\phi$ and since $G\phi \in L_\Delta$ we get that $G\phi \in \Delta$!). So $\vdash G(\wedge \psi_i \to \phi)$.

Now since for each i $G\psi_i \in \Delta \cap L_{\Delta\phi}$ we may conclude (by the definition of $L_{\Delta\phi}$ and by axioms (2) and (3)) that $Tp_j \in \Delta$, for each p_j that occurs in ψ_i and not in ϕ.

Since $\vdash \wedge Tp_j \to [G(\wedge \psi_i \to \phi) \to (G \wedge \psi_i \to G\phi)]$ we get that $G(\wedge \psi_i \to \phi)$

$\rightarrow (G \wedge \psi_i \rightarrow G\phi) \in \Delta$ and so since the antecedent is provable we get that $G \wedge \psi_i \rightarrow G\phi \in \Delta$ and since $\vdash \wedge G\psi_i \rightarrow G \wedge \psi_i \in \Delta$ we get that $G\phi \in \Delta$, which is a contradiction.

This was essentially Rennie's argument. We now extend Δ_0 to a complete theory Δ^{ϕ} in the language $L_{\Delta^{\phi}}$.

LEMMA 26. $L_{\Delta} \supseteq L_{\Delta^{\phi}}$.

Proof. This is true by the definition of $L_{\Delta^{\phi}}$ since $G\phi \in \Delta$ and $G\psi \in \Delta$ imply that ϕ and $\psi \in L_{\Delta}$.

Lemma 27. *Let Δ be a complete and consistent SK_t theory and let Δ^{ϕ} be constructed as in* (25) *then if $T\psi \in \Delta$ then $\psi \in L_{\Delta^{\phi}}$.*

PROOF. By construction.

LEMMA 28 Rennie [5]. *Let Δ be a consistent RQ_t theory and let $\sim T\phi \in \Delta$. Let $\Delta_0 = \{\psi \mid G\psi \in \Delta \cap L_{\Delta(\phi)}\}$ where $L_{\Delta(\phi)}$ is the language built up from all propositional variables appearing in* any *$T\beta \in \Delta$, then Δ_0 is RQ_t consistent and $\phi \in L_{\Delta(\phi)}$.*

PROOF. This was proved by Rennie.

(29) Since Δ_0 is RQ_t consistent it can be extended to a complete and consistent RQ_t theory $\Delta^{(\phi)}$. We have that $L_{\Delta} \supseteq L_{\Delta(\phi)}$.

(30) When we deal with the logic SK_t and not with an extension, we proceed as follows:

Let Δ be a consistent SK_t theory and let $\sim T\phi \in \Delta$ let $L_{\Delta(\phi)}$ be the language built up from the propositional variables that appear in any $T\phi$ for $T\psi \in \Delta$. Let $\Delta^{(\phi)}$ be any complete and SK_t consistent theory in this language. We have $L_{\Delta} \supseteq L_{\Delta^{\phi}}$, the proof as in the previous case.

We can similarly construct theories Δ_{ϕ} for $\sim H\phi \in \Delta$ and $\Delta_{(\phi)}$ for $\sim Y\phi \in \Delta$, for both cases, Δ being in SK_t theory and Δ being an RQ_t theory.

(31) We are in a position to construct, for a given theory Δ (either SK_t or QK_t etc. theory) a model W which will serve us in the proof of the completeness theorem. In the sequel we shall speak about consistent theories, it being understood that all the theories are in some system e.g. SK_t or QK_t etc.

The elements of W are finite sequences of elements of the form $\bar{\phi}$ or ϕ or $(\bar{\phi})$ or (ϕ) where ϕ is a sentence of the language. With each $x \in W$ there will be associated a language L_x and a complete and consistent theory (in the appropriate logic) $\Delta(x)$.

Let Δ be given. Let $0 \in W$ where 0 is the empty sequence and let $L_0 = L_\Delta$ and $\Delta(0) = \Delta$. We continue our construction by induction. Suppose that for all sequences x of length $\leqq n$ we know whether $x \in W$ or not and that in case $x \in W$ L_x and $\Delta(x)$ have been defined. We now give the definitions for sequences of length $n + 1$. Let $x \in W$ of length n be given. For every ϕ such that $\sim G\phi$ is in $\Delta(x)$ we construct a theory $(\Delta(x))^\phi$ and the language $L_{(\Delta(x))^\phi}$. We let $y = x^*\langle\bar{\phi}\rangle \in W$ and let $\Delta(y) = (\Delta(x))^\phi$ and $L_y = L_{(\Delta(x))^\phi}$ (see 25). (Where $*$ is concatenation of sequences).

Similarly for $\sim T\phi \in \Delta(x)$ we form $y = x^*\langle(\bar{\phi})\rangle \in W$ and L_y and $\Delta(y) = (\Delta(x))^{(\phi)}$ (see 29, 30).

For the cases of $\sim H\phi$ and $\sim Y\phi \in \Delta(x)$ we form $x^*\langle\underline{\phi}\rangle$ and we let $\Delta(x^*\langle\underline{\phi}\rangle) = (\Delta(x))_\phi$ and similarly for $x^*\langle(\underline{\phi})\rangle$.

Let W be the set of all sequences thus defined. Let L_x be the respective languages. Let A_x be defined as follows: for $p \in L_x$ let $[p]_x = 1$ iff $p \in \Delta(x)$ (p a propositional variable).

We now define $<$, $>$, $R_<$, $R_>$ as follows:

(32) $x < y$ = iff $y = x^*\langle(\bar{\phi})\rangle$ for some ϕ.

$x > y$ iff $y = x^*\langle(\underline{\phi})\rangle$ for some ϕ.

$xR_<y$ iff $y = x^*\langle\bar{\phi}\rangle$ for some ϕ.

$xR_>y$ iff $y = x^*\langle\underline{\phi}\rangle$ for some ϕ.

LEMMA 33.

(a) $x\rho^*y \Rightarrow L_x \supseteq L_y$.

(b) $G\phi \in \Delta(x) \wedge xR_<y \Rightarrow [\phi \in L_y \Rightarrow \phi \in \Delta(y)]$.

(c) $T\phi \in \Delta(x) \wedge (xR_<y \vee x < y) \Rightarrow \phi \in L_y$.

(d) The analogues of (b) and (c) for H and Y.

(e) In the case that Δ was a RQ_t theory then $G\phi \in \Delta(x)$ and $x < y$ and $\phi \in L_y \Rightarrow \phi \in \Delta(y)$. Similarly for H.

PROOF. By construction; for (e) see (28).

THEOREM 34. *SK_t is complete for the semantics described in* (17).

Proof. One can easily verify that all SK_t theorems hold in this semantics. Let Δ be a complete and consistent SK_t theory, we shall show it has a model. Let W be the structure constructed above for Δ. We prove the following:

(35) $[\phi]_x = 1$ iff $\phi \in \Delta(x)$; for $\phi \in L_x$, and the appropriate R_G, R_H, $<_T$, $>_Y$ see (17).

For propositional ϕ this holds by definition.

Conjunction and negation present no difficulties.

Assume that $T\phi \in \Delta(x)$, then by (33), ϕ is defined in all worlds y such that $x < y$ or $xR_{<}y$. Now suppose that $yR_{>}y$; this means that $L_y \supseteq L_x$ and so certainly ϕ is defined at y and so we see that ϕ is defined in all y such that $x\rho_{<}y$. Conversely if $\sim T\phi \in (x)$ then ϕ is not defined in y where $y = x^*\langle(\bar{\phi})\rangle$.

Assume that $G\phi \in \Delta(x)$. Let $xR_{<}y$ then if ϕ is defined at y this means that $\phi \in \Delta(y)$ by the construction of any $(\Delta(x))^{\Psi}$ (see Lemma 33). Now let $yR_{>}x$, this means that for some ψ $\Delta(x) = (\Delta(y))^{\Psi}$. In this case ϕ is certainly defined in L_y since $L_y \supseteq L_x$ and ϕ is defined in L_x. Now if $\sim \phi \in \Delta(y)$ then by the axioms $H \sim G\phi \in \Delta(y)$ and so by (33) for the case of H we get that $\sim G\phi \in \Delta(x)$ which is a contradiction. This shows that if $G\phi \in \Delta(x)$ then $\phi \in \Delta(y)$ for all y such that xRy. Conversely if $\sim G\phi \in \Delta(x)$ then $\sim \phi \in \Delta(y)$ for $y = x^*\langle\bar{\phi}\rangle$.

The cases of $Y\phi$ and $H\phi$ can be treated similarly. This concludes the proof of (35).

THEOREM 36. *QK_t is complete for the semantics described in* (18).

PROOF. First let us show that the following holds:

(37)$xR_Gy \wedge y <_T z \Rightarrow x <_T z$

$xR_Hy \wedge y >_T z \Rightarrow x >_T z.$

To prove this recall that by (18a) and (24) we have to show that $xRy \wedge \exists u (y\bar{R}u \wedge u\rho_{<}z) \Rightarrow \exists v(x\bar{R}v \wedge v\rho_{<}z)$

which is valid. Similarly the other implication holds.

Let us now show that (7) holds in this semantics. Let $[\sim T\phi]_x = 1$ and $[H \sim T\phi]_x = 0$.

Then for some y such that $x <_T y$ we have that ϕ is not defined in y, and for some z such that zRx we have that $[T\phi]_z = 1$. Now since we have $zRx \wedge x <_T y$ we get that $z <_T y$ which is a contradiction. Similarly the other axiom holds.

Let Δ be a complete and consistent QK_t theory. We want to show that Δ has a model in this semantics. To get this we repeat the construction of W as in (31) and we have to show that (35) holds for the appropriate R_G, R_H, $<_T$, $>_Y$ (see 18). To show this all we have to show that if $[T\phi]_x = 1$ and $x <_T y$ then ϕ is defined at y (i.e. $\phi \in L_y$). This is proved by induction on the length of the sequence $y_1 \cdots y_n$ leading from x to y. (See (24), recall that $\bar{R}$ is the transitive closure of R). Suppose that xRy_1 and ($y_1 < y$ or y_1Ry) and that $T\phi \in \Delta(x)$. The latter implies that ϕ is defined at y_1. If ϕ is not defined at y then $\sim T\phi \in \Delta(y_1)$ and so by the axiom, $H \sim T\phi \in \Delta(y_1)$ and since xRy_1 holds we get that $\sim T\phi \in \Delta(x)$ which is impossible.

Now assume that xRy_0 and that $y_0R^{n-1}u$ and ($u < y$ or uRy) hold. Let $T\phi \in \Delta(x)$ and assume that ϕ is not defined at $\Delta(y)$. Then by the induction hypothesis $\sim T\phi \in \Delta(y_0)$ and so $H \sim T\phi \in \Delta(y_0)$ and so $\sim T\phi \in \Delta(x)$ which is a contradiction. Similar lemma holds for the case of $>_Y$. This completes the proof of the completeness theorem for QK_t.

THEOREM 38 *Rennie* [5]. *RQ_t is complete for the semantics described in* (19).

PROOF. One can easily verify that all the theorems of RQ_t hold. To get completeness we construct, for a given theory Δ the model W. To prove (35) for our case we use (33e).

We are now in a position to solve an open problem of Rennie [5], namely the semantics for QCR.

THEOREM 39. *QCR is complete for the semantics described in* (20).

PROOF. Clearly, for transitive $\bar{\rho}$ the axiom holds.

Let Δ be a consistent and complete QCR theory. We construct the model W (bearing in mind that $RQ_t \subseteq QCR$). To get completeness we have to show that (35) holds for $\bar{\rho}$. For this we recall lemma 23. To conclude the proof of (35) we have to show that if $x\bar{\rho}y$ and $G\phi \in \Delta(x) \cap L_y$ then $\phi \in \Delta(y)$ and similarly for $T\phi$, $H\phi$, and $Y\phi$. It is sufficient to show that if $x\rho y\rho z$ and $G\phi \in \Delta(x) \cap L_z$ then $\phi \in \Delta(z)$ and similarly for the case of $T\psi \in \Delta(x)$.

Assume that $\sim \phi \in \Delta(z)$ then since $\phi \in L_x \cap L_z$ by theorem (23) $\phi \in L_y$ and so since $\vdash_{QCR} G\phi \to GG\phi$ we get that $GG\phi \in \Delta(x)$ and so $G\phi \in \Delta(y)$ and therefore $\phi \in \Delta(z)$.

Now assume that $T\psi \in \Delta(x)$ and that ψ is undefined at L_z. So $\sim T\psi \in \Delta(y)$ and hence $H \sim T\psi \in \Delta(y)$ and therefore $\sim T\psi \in \Delta(x)$ which is a contradiction. This completes the proof of (39).

2. The system S_t

In this section we shall describe a basic system with a statability operator $S\,\phi$ in which all other systems are faithfully interpretable. Our language contains besides the symbols for Lemmon's K_t the unary operator $S\,\phi$ which reads: ϕ is statable right now. We have the following axioms:

(40) Axioms (1), (4) and (5) of SK_t (see section 1).

(41) $S\phi \to Sp$ where p is a propositional variable occurring in ϕ.

(42) $Sp_1 \wedge \cdots \wedge Sp_n \to S\phi$ where $p_1 \cdots p_n$ are all the propositional variables occurring in ϕ.

(43)
$$G(\phi \to \psi) \to (G\phi \to G\psi)$$
$$H(\phi \to \psi) \to (H\phi \to H\psi).$$

Let us now describe the semantics for which S_t is complete. An S_t structure is a system $(A_x, L_x, R, 0)$ $0 \in W$, $x \in W$ where R is a binary relation on W and A_x, $x \in W$ is a classical propositional model in the language $\bigcup_x L_x$. The truth value of the operators is defined as follows:

(45) ϕ is stable at x iff $\phi \in L_x$.
$[S\phi]_x = 1$ iff $\phi \in L_x$.
For statable $G\phi$ and $H\phi$ we define:
$[G\phi]_x = 1$ iff $\forall y(xRy \Rightarrow [\phi]_y = 1)$
$[H\phi]_x = 1$ iff $\forall y(yRx \Rightarrow [\phi]_y = 1)$.

THEOREM 46.
S_t is complete for the above semantics.

PROOF. It is easy to verify that all the axioms are valid.

Let Δ be a consistent and complete S_t theory. We want to define a model of Δ. To do this we can construct a model W like the one we constructed in section 1. We shall not go through the details but just give the crucial lemma.

LEMMA 47. *Let Δ be a complete and consistent theory and let $\sim G\phi \in \Delta$. Let Δ_0 be $\{\sim \phi\} \bigcup \{\psi \mid G\Psi \in \Delta\}$, then Δ_0 is consistent.*

PROOF. Otherwise we have that for some $\psi_1 \cdots \psi_n$: $\vdash \psi_1 \wedge \cdots \wedge \psi_n \to \phi$ so $\vdash G(\bigwedge \psi_i \to \phi)$, and so $\vdash \bigwedge G\psi_i \to G\phi$.

We therefore conclude that $G\phi \in \Delta$ which is a contradiction.

A similar lemma holds for the case of H.

Using these two lemmas we can carry out the proofs of the completeness theorem, along the lines of section 1. The only addition we need is that the language associated with a theory $\Delta(x)$ (see [31]) is the set of all ϕ such that $S\phi \in \Delta(x)$. The axioms on S ensure that this is a good definition.

The system RQ_t may be interpreted in S_t as follows:

(48) $(p)_* = p$ for a propositional variable p.

$$(\phi \wedge \psi)_* = (\phi)_* \wedge (\psi)_*$$
$$(\sim \phi)_* = \sim (\phi)_*$$
$$(G\phi)_* = G(S\phi_* \rightarrow \phi_*)$$
$$(H\phi)_* = H(S\phi_* \rightarrow \phi_*)$$
$$(T\phi)_* = GS\phi_*$$
$$(Y\phi)_* = HS\phi_*.$$

We thus see that in fact S_t is more basic.

We now turn to consider modal systems with statability operators. Such systems were considered by Bull [3] and Segerberg [6]. Both added to the usual modality operator some statability operator. I would like to show how a system K_s may be constructed with just one modal operator G which has the properties of a statability operator. First let us describe the semantics of this system.

A structure is a system $(A_x, L_x, R, 0)$ $x \in W$, $0 \in W$ where A_x is a classical model n the language L_x and R is a binary relation on W, the definition of the truth table for G is:

(49) $[G\phi]_x = 1$ iff for all y such that xRy and ϕ is defined at y we have $[\phi]_y = 1$, and $G\phi$ is defined at x.

As axioms for this logic we may take, in addition to the axioms and rules of the classical propositional calculus, the following axioms.

(50) $\vdash \phi \Rightarrow \vdash G\phi$.

(51) $G(\phi \rightarrow \psi) \rightarrow (G\phi \rightarrow G\psi)$ whenever every propositional variable that occurs in ϕ occurs also in ψ.

To prove completeness we need the following result:

(52) Let Δ be a complete and consistent theory and let $\sim G\psi \in \Delta$. Let $L_{\Delta\psi}$ be the language built up from all the propositional variables that occur in ψ, then $\{\phi \mid G\phi \in \Delta \cap L_{\Delta\psi}\} \bigcup \{\sim \psi\}$ is consistent.

PROOF. Otherwise for some ϕ_i

$\vdash \bigwedge \phi_i \rightarrow \psi$

$\vdash G(\bigwedge \phi_i \rightarrow \psi)$

$\vdash G(\phi_i \rightarrow (\phi_2 \rightarrow \cdots \rightarrow \psi) \cdots)$

and so $\vdash G\phi_1 \rightarrow (G\phi_2 \rightarrow \cdots \rightarrow G\psi) \cdots)$

and so $G\psi \in \Delta$ which is a contradiction

3. Decidability results

Rather than repeat all the constructions in part 1 [2] we shall assume that the reader is familiar with [4] and [2]. From the results of [4] it follows that (53) holds.

THEOREM 53 (Rabin). *The monadic second order theory of a denumerable system of the form* $(W, <, >, R_<, R_>, \bar{R}_<, \bar{R}_>, \rho_<, \rho_>, \bar{\rho}_<, \bar{\rho}_>, 0)$ *is decidable.* Where W is a tree as in section 1.

From the results in [2] it follows that any system whose semantics has an accessibility relation R_G, R_H. $<_T$, $>_Y$ that can be expressed in the above monadic 2nd order theory and is finitely many valued is decidable. Now by theorem (16) we see that systems have this property. Each propositional variable has as assignment a pair of subsets of W, the set of points at which it is true, and the set of points at which it is false respectively. The complement of the union of these sets is the set of all points at which it is undefined.

We therefore conclude.

THEOREM 54 *Decidability Theorem. The systems* SK_t, RK_t, QK_t, QCR, S_t *and the other systems of* (16) *are decidable and so is* K_s.

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THE HEBREW UNIVERSITY OF JERUSALEM